1 DISCLAIMER

These notes are still in heavy revision, and I don't guarantee that what's here is correct or, regarding the computational models, altogether what I think we should use.

2 Concept of effective stress

Hooke's law for a 1D saturated porous medium with pore pressure effects included reads

$$\sigma_s + \alpha(p - p_r) = \gamma\epsilon_s \tag{1}$$

where α is the Biot constant (which we hereafter set its usual value of unity), σ is the stress, ϵ is the strain, γ is a constant that depends on elastic properties, p is the pressure, the subscript s refers to the soil matrix, and the subscript r refers to a reference value. We adopt the convention that the stress is negative for compression. Define the amount of pore pressure p^* in excess of the hydrostatic pressure (hereafter called the excess pressure) as

$$p^* = p - (p_{surf} + \rho_w g z) \tag{2}$$

where ρ is the density, g is the gravitational acceleration, z is the depth, the subscript w refers to water, and the subscript $surf$ refers to the surface value. Setting p_r to p_{surf} in (1) and combining with (2) yields

$$\sigma_s + (p^* + \rho_w g z) = \gamma\epsilon_s \tag{3}$$

Hence, for any deformation to occur in the soil matrix the stress must exceed the value $-(p^* + \rho_w g z)$. We define the effective stress σ'_s as the amount of stress in excess of this quantity, so that we may write

$$\sigma'_s \equiv \sigma_s + (p^* + \rho_w g z) = \gamma\epsilon_s. \tag{4}$$

3 Consolidation under surface loading

Consider a 1D column of fluid-saturated porous medium; let σ_l be the force per unit area supplied by a load at the top of this column. This force will contribute to the deformation of the soil matrix and to an increase in fluid pore pressure; therefore, we may write

$$\sigma_l = -\sigma'_s + p^*. \tag{5}$$

Now suppose that the effect of the surface load is to compress the soil matrix while leaving the total volume of soil unchanged. Even though the soil volume is unchanged, the soil will occupy a smaller volume of space than before because of the decreased void space. The decrease in the volume of space occupied is given as

$$-\Delta V = -\beta V_0 \Delta\sigma'_s \tag{6}$$

where β is the soil matrix compressibility, V is the column volume, and the subscript naught refers to the initial value. $-\Delta V$ is also the decrease in space available for fluid to occupy, that is, the decrease in the void space volume. Hence,

$$-\Delta V = -\Delta V_v = V_{v0} - V_v = \phi_0 V_0 - \phi V \approx -V_0 \Delta\phi \tag{7}$$

where the subscript v refers to the void space, and the approximation holds good as long as $-\Delta V$ is not too large. Combining equations (7) and (6), dividing by the time increment Δt, and letting this increment go to zero yields

$$\beta \frac{\partial \sigma_s'}{\partial t} = \frac{\partial \phi}{\partial t}. \tag{8}$$

The decrease in space available to the fluid will cause the pore pressure to increase, which will in turn drive fluid from the top or the bottom of the column. The equation governing the pressure evolution during this process can be obtained by substituting (8) into the mass continuity equation. The substitution gives

$$\beta \frac{\partial \sigma_s'}{\partial t} = -\frac{\partial v}{\partial z} \tag{9}$$

where v is the vertical Darcy velocity, which may be written in terms of the excess pressure p^* as

$$v = -\frac{k}{\mu} \frac{\partial p^*}{\partial z}. \tag{10}$$

In this expression, k is the permeability and μ is the dynamic viscosity. Putting (10) into (9) and using (5) to evaluate the time derivative results in

$$\frac{\partial p^*}{\partial t} = \frac{k}{\beta\mu} \frac{\partial^2 p^*}{\partial z^2}. \tag{11}$$

where we have assumed that σ_l is constant in time. Equation (11) is the fundamental equation of consolidation theory as advanced by K. Terzaghi. The combination $k/\beta\mu$ we denote as c_v and refer to as the consolidation constant. For a 1D problem we set poisson's ratio ν equal to zero; then may also write $c_v = Ek/\mu$ where E is Young's modulus.

4 Subsidence due to thawing

Consider a 1D column of fully-saturated frozen soil heated from above, so that a thaw front progresses from the surface downward. The depth of this thaw front after a time t has passed is given by

$$Z(t) = (\kappa_{eff} t)^{\frac{1}{2}} \tag{12}$$

where κ_{eff} is an effective diffusivity constant. When a thin layer of height ΔZ thaws, the soil will deform under it's own weight as well as that of the soil above it. Let us assume, as in section 3, that this deformation results in decreased porosity without decreasing the soil volume. Then equation (11) applies and we may used a fixed pressure condition at the top boundary. In order to formulate a well-posed problem, therefore, it remains only to find the appropriate boundary condition for the thaw front location $z = Z(t)$. The soil below this depth does not deform as is hence of no interest. If the average Darcy velocity of the fluid in the layer ΔZ is v then the volume of fluid liberated from the top of this layer in an increment of time Δt is

$$-\Delta V = -vA\Delta t = \frac{kA\Delta t}{\mu}\frac{\partial p^*}{\partial z} \tag{13}$$

where V is the volume of the layer and A is the cross-sectional area. The minus sign in the middle term reflects that we have chosen v as negative upward. Then the fractional volume decrease is

$$-\frac{\Delta V}{V} = \frac{k}{\mu}\frac{\Delta t}{\Delta Z}\frac{\partial p^*}{\partial z} = \beta\Delta\sigma'_s \tag{14}$$

where we have used the fact that $A/V = 1/\Delta Z$ for the first equality and equation (6) for the second. Assuming that there is no deformation initially, $\sigma'_{s0} = 0$. Also, by definition (4) and the fact that $\sigma_s = -\rho_s g Z$ we have

$$\sigma'_s = -\rho_s g Z + p^* + \rho_w g Z = p^* - \Delta\rho g Z \tag{15}$$

where $\Delta\rho \equiv \rho_s - \rho_w$. Equation (15) says that deformation of the soil matrix is caused by a combination of the excess pore pressure and the so-called "floating weight" of the soil itself. Putting (15) into the second equation of (14) and taking the limit as $\Delta Z \to 0$ gives

$$p^* - \Delta\rho g Z = c_v\left(\frac{dZ}{dt}\right)^{-1}\frac{\partial p^*}{\partial z} \tag{16}$$

where p^* and its derivative are to be taken at $z = Z$. Equation (16) is the boundary condition that p^* must satisfy at $z = Z$. Once (11) has been solved with the boundary condition (16), equation (3) may be integrated to find the soil displacement.

5 Subsidence due to drainage

Consider again a 1D column of fully-saturated frozen soil heated from above, with a thaw front progressing downward. Furthermore, let there be a horizontal drainage face at the bottom, i.e., a horizontal fluid flow pathway leading to some region of pressure lower than the hydrostatic pressure at that depth. We will without loss of generality consider the case when the drainage face leads to the surface at some lateral distance from the column. This situation might describe, for example, a horizontal fracture at the bottom of a column of soil on top of a hill. Fluid will begin to exit the column from the bottom at the moment the thaw zone reaches the drainage face. We will assume that fluid vacates to some extent the pore space and that the soil is sufficiently weak as to collapse into the newly created void instantaneously. During this process, surrounding pressures will approach the pressure, p_{surf}, of the drainage face. Consider a thin section of height ΔZ in which the soil has just begun to fail. By combining equations (7) and (13) we obtain

$$\frac{k}{\mu}\frac{\partial p^*}{\partial z}A\Delta t \approx V\Delta\phi. \tag{17}$$

Dividing both sides by V and letting Δt go to zero gives

$$d\phi = \frac{k}{\mu}\frac{\partial p^*}{\partial z}\left(\frac{dZ}{dt}\right)^{-1} \tag{18}$$

where dZ/dt is the average upward velocity of the drainage zone, i.e., of the pressure pulse that ensues at the onset of drainage. Averaging both sides of (18) across the layer yields

$$\phi = \phi_0 + \frac{k}{\mu \Delta Z} \left(\frac{dZ}{dt} \right)^{-1} (p^* + \rho_w g Z), \tag{19}$$

where we have used the fact that $p_0^* = -\rho_w g Z$ and assumed that the combination $\Delta Z(dZ/dt)$ is constant. Because $p^* + \rho_w g Z = p - p_{surf}$, equation (19) is equivalent to

$$\phi = \phi_0 + \alpha(p - p_{surf}), \tag{20}$$

where

$$\alpha \equiv \frac{k}{\mu \Delta Z} \left(\frac{dZ}{dt} \right)^{-1}. \tag{21}$$

Equation (20) is the starting point for the paper on subsidence that we have recently completed. We may estimate the velocity of the pressure pulse graphically from an analytic solution obtained on substituing (20) into the continuity equation and solving with the appropriate boundary conditions. This velocity may then be compared with the value given by equation (21) with the parameter values used to obtain the analytic solution plugged into it; the velocities obtained in this way are found to agree.

6 Computational framework for calculating consolidation under small strains

Consider a control volume of initial height h_0 with node location z_0, and take the z coordinate as positive downward. If the control volume is embedded in the soil matrix, then a deformation $u(z)$ in the soil matrix will lead to a corresponding deformation of the control volume. The top of the control volume will be displaced by an amount $u(z_0 - h_0/2)$ while the bottom will be displaced by $u(z_0 + h_0/2)$, where u must be positive downward because z is positive downward. Then the height after deformation of the control volume is given as

$$h = h_0 + u(z_0 + h_0/2) - u(z_0 - h_0/2) \approx h_0 \left(1 + \frac{\partial u}{\partial z} \right) \tag{22}$$

and hence the change in height is

$$\Delta h \approx h_0 \frac{\partial u}{\partial z}. \tag{23}$$

If the cross-sectional area of the control volume does not change, then equation (7) implies that

$$\Delta h \approx h_0 \Delta \phi. \tag{24}$$

Combining the last two equations gives

$$\Delta \phi \approx \frac{\partial u}{\partial z} \equiv \epsilon \tag{25}$$

where ϵ is the vertical strain. If equation (1) is generalized to include thermal effects the result is

$$\sigma_s + \alpha_1(p - p_r) + \alpha_2(T - T_r) = \gamma\epsilon_s. \tag{26}$$

The force balance equation governing the stress σ_s is

$$\frac{\partial \sigma_s}{\partial z} = -\rho_s g. \tag{27}$$

Putting (25) into (26) and substituting the result into (27) yields

$$\frac{\partial(\gamma\phi)}{\partial z} = \alpha_1 \frac{\partial p}{\partial z} + \frac{\partial(\alpha_2 T)}{\partial z} - \rho_s g + \phi_0 \frac{\partial \gamma}{\partial z}. \tag{28}$$

Integrating the above equation from the surface downward yields

$$\phi = \frac{1}{E}\left[(p - p_0) + \alpha T - \alpha_0 T_0 - \rho_s g z + \phi_0 E\right] \tag{29}$$

where the naught subscript refers to surface values, α_1 has been set to unity, α_2 has been replaced with α, and γ has been replaced with E. When ice is present, the loss of volume in soil matrix compression is not equal to the entire void space volume, but only to that portion occupied by liquid or gas phases. In this case, define

$$\tilde{\phi} = \phi(1 - S_i) \tag{30}$$

where S_i is the volume fraction of ice. The derivation of (29), with ϕ at each step replaced by $\tilde{\phi}$, is unaltered. Therefore, in terms of ϕ equation (29) becomes

$$\phi = \frac{1}{(1 - S_i)E}\left[(p - p_0) + \alpha T - \alpha_0 T_0 - \rho_s g z + \phi_0(1 - S_{i0})E\right]. \tag{31}$$

The strength of the soil E is a function of the liquid, gas, and ice volume saturations $S_{l,g,i}$ as well as the initial porosity ϕ_0. Similarly, the thermal expansion coefficient α is a function of these variables. Equation (31) may be coupled together with differential equations representing mass and energy conservation to obtain a set of three differential equations in three unknowns. These equations can then be solved by a some nonlinear implicit numerical procedure, i.e., Newton-Raphson iterations, to obtain T, p, and ϕ. From the decrease in porosity, (24) may be used to calculate the new height of a control volume, and then surface subsidence may be obtained by adding up the height decrease along each vertical column of control volumes.

7 Fully saturated column

As an illustration of how equation (31) predicts thaw subsidence consider a fully saturated, thawed column of height L. For simplicity suppose that αT does not vary and that $S_{i0} = 0$. Then (31) becomes

$$\phi = \phi_0 - \frac{\Delta\rho g z}{E} \tag{32}$$

where $\Delta\rho = (\rho_s - \rho_l)$. The average decrease in porosity of the column is

$$-\langle\Delta\phi\rangle = \int_0^L \frac{\Delta\rho g z}{LE} dz = \frac{\Delta\rho g L}{2E}. \tag{33}$$

The fractional decrease in the height of the column is related to $\langle\Delta\phi\rangle$ by

$$\mathcal{F} \equiv \frac{\Delta L}{L} = \langle\Delta\phi\rangle. \tag{34}$$

Hence, for a given value of $\mathcal{F}$,

$$E = -\frac{\Delta\rho g L}{2\mathcal{F}}. \tag{35}$$

For an example of the magnitude of E that this formula implies, the values $\mathcal{F}$ = 0.3, $\Delta\rho$ = 700 kg/m^3, g = 9.8 m/s^2, and L = 10 m give E = 0.11 MPa.

8 Partially saturated column

9 Massive ice

Consider a volume cell of mostly ice with a small amount of soil dispersed within it. The soil displacement during melting of the ice cannot be modeled via Hooke's law because the soil strain will now be large. If the volume of the cell is taken as the smallest cube that encloses the soil, that volume will decrease upon melting of the ice. The porosity is related to the cell volume by

$$V = \frac{V_v}{\phi} = \frac{V - \frac{M_s}{\rho_s}}{\phi} \tag{36}$$

where M_s is the mass of soil in the cell. Solving this equation for V one obtains

$$V = \frac{M_s}{\rho_s(1-\phi)} \tag{37}$$

and solving it for ϕ yields

$$\phi = 1 - \frac{M_s}{V\rho_s}. \tag{38}$$

We seek a formula that gives V as a function of ice volume saturation S_i. The porosity will then also be given as a function of S_i via equation (38). The function $V = V(S_i)$ should satisfy

1. $V(S_i = 0) = V_{thw}$
2. $V(S_i = S_{i,0}) = V_0$
3. $\frac{dV}{dS_i}|_{S_i=S_{i,crt}} >> 0$

where V_thw is the volume of the cell after it has been completely thawed and $S_{i,crt}$ is a critical "collapse" ice saturation below which the cell's volume decreases rapidly upon further melting of the ice. Using (37), V_{thw} is given in terms of M_s, ρ_s, and some minimum allowed porosity ϕ_{thw} by

$$V_{thw} = \frac{M_s}{\rho_s(1-\phi_{thw})}. \tag{39}$$

A function that satisfies the above criteria is given as

$$V(S_i) = Atanh[B(S_i - S_{i,crt})] + C \tag{40}$$

where B is some large number that controls the slope of $V(S_i)$ at $S_{i,crt}$,

$$A = \frac{V_0 - V_{thw}}{tanh[B(S_{i,0} - S_{i,crt})] + tanh(BS_{i,crt})} \tag{41}$$

and

$$C = V_{thw} + Atanh(BS_{i,crt}). \tag{42}$$